Research Statement

Justin M. Troyka

My research is in enumerative and algebraic combinatorics. I study combinatorial objects such as permutations, partitions, and graphs, often using tools from algebra, analysis, and probability. My favorite questions in this area are asymptotic: for instance, I have proved asymptotic results about the number of cyclic permutations with a given descent set [3], and I have proved a conjecture on the asymptotic number of split graphs that are *balanced* [10]. I also do work in permutation patterns, especially on growth rates of permutation classes. I have published two papers [6, 7] in the last year and have two more [4, 11] in the review process. Parts of my research would be excellent for collaboration with undergraduates, and I would love to work with students on these projects that I am passionate about. My research is related to interdisciplinary work on analysis of time-series data [2], such as the height of ocean tides or stock market performance over time; I would be excited to advise a student research project on these topics.

Permutation patterns. Figure 1 illustrates one permutation contained in another, meaning the former permutation has the same relative order as some subsequence of the latter permutation. A *permutation class* is a set $\mathcal{C}$ of permutations such that every permutation contained in an element of $\mathcal{C}$ is also in $\mathcal{C}$. These concepts are central to the research area of permutation patterns, a beautiful and multifaceted subfield of combinatorics where sequences like the Catalan numbers and the Fibonacci numbers often arise. In the last decade or so, there has been growing interest in understanding the set of all exponential growth rates of permutation classes. These efforts have begun to uncover an intricate landscape.

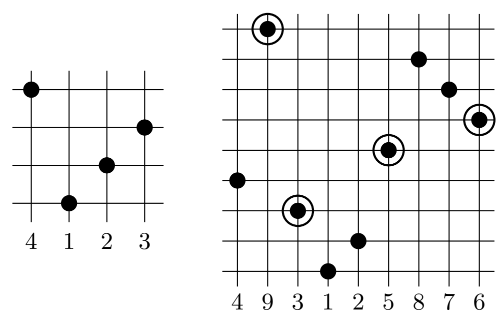

Figure 1: The permutation 4123 is contained in the permutation 493125876.

In much of my research, I adapt the field's fundamental questions to other objects related to permutations, using tools from combinatorics, algebra, probability, and analysis to understand how this new landscape compares to the well-studied landscape from "classical" permutation patterns. Part of my PhD thesis was on patterns in centrosymmetric permutations (meaning the diagram is fixed by half-turn rotation), finding general types of classes with the same growth rates as their classical counterparts [9]. Neal Madras and I have written two papers [6, 7] on patterns in *affine permutations*, a generalization of permutations in which the entries repeat infinitely. Jonathan Fang, Zachary Hamaker, and I [4] studied a new type of pattern in *matchings*, which we view as permutations consisting of 1-cycles and 2-cycles. An intriguing finding from [4] which requires further study: whereas a classical permutation class must have finite exponential growth rate, for matchings there is a dichotomy between finite exponential growth rates and factorial growth.

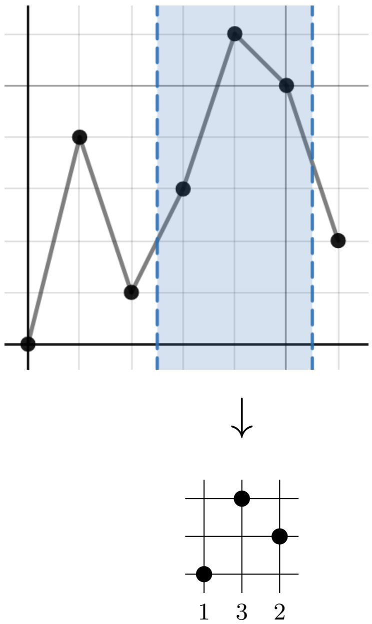

Figure 2: A permutation pattern contained in time series data. As the window moves from left to right and generates a pattern in each position, we can statistically analyze the frequencies of the various patterns.

There are many deep questions that point to the development of a comprehensive theory, with seemingly simple problems that have been open for decades, the larger picture just starting to be seen. Still, aspects of my research in permutation patterns would be great to share with undergraduates, as there are many fun problems where students can start right away and really dig in. The undergraduates I have worked with include Luke Seaton, with whom I worked during Summer 2021 on new directions for the affine permutations project mentioned above, and Jonathan Fang, one of the collaborators on the involutions project. Another area of permutation patterns that would be especially ripe for student research is the analysis of time-series data using permutation entropy [2] (as shown in Figure 2), a topic in statistical physics which has great potential for interdisciplinary work with the sciences and economics. A student who works in permutation patterns with me would likely present a poster or talk at the *Permutation Patterns* international conference.

Enumeration of cyclic permutations according to descent set. This section covers my thesis work on descent sets of cyclic permutations, which is published in a joint paper [3] with my PhD advisor, Sergi Elizalde. What fascinates me about this topic is the peculiar interplay between two permutation concepts: the permutation as a sequence (*one-line notation*) and the cycle structure of the permutation. I will also explain how I am now building on that research and plan to continue doing so.

Given a permutation π, the *descent set* of π is the set of positions i where $\pi(i) > \pi(i+1)$. We are concerned with the descent sets of permutations whose cycle type is that of a full cycle, as shown in Figure 3. Gessel & Reutenauer [5] express the number of size-n permutations with given descent set and given cycle type as the scalar product of certain symmetric functions; our contribution is to extend that result, translating it into an explicit combinatorial formula counting the n-cycles with given descent set:

▶▶▶ **Theorem 1.** (Elizalde & Troyka [3]) *For any set of positions I, the number of cycles of size n with descent set I is $\frac{1}{n}\sum_{d\,|\,n}\mu(d)\,(-1)^{|I|-|I/d|}\,\beta_{n/d}(I/d)$, where $\beta_k(I)$ denotes the total number of size-k permutations with descent set I, μ is the Möbius function from number theory, and $I/d = \{i/d \colon i \in I \text{ and } d \mid i\}$.*

From Theorem 1 we prove several new results, of which perhaps the most striking is the following asymptotic result, generalizing Stanley's [8] remark that "the properties of being an alternating permutation and an n-cycle are 'asymptotically independent'":

▶▶▶ **Theorem 2.** (Elizalde & Troyka [3]) *For almost all descent sets I, the proportion of cycles among permutations with descent set I is asymptotically the same as the proportion of cycles among all permutations.*

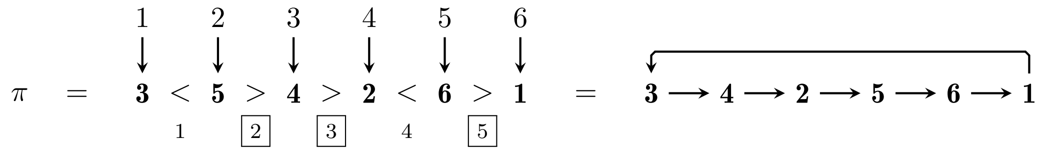

Figure 3: The permutation $\pi = \mathbf{354261}$, first as a sequence with its descents shown, then as a cycle. The descent set of π is $\{2,3,5\}$, and the cycle type of π is that of a full 6-cycle.

In other words, the properties of having descent set I and being an n-cycle are asymptotically independent, for almost all I. In fact, I am in the midst of finding a proof that this holds for *all* non-trivial I, which will soon be ready to write up and submit. Our original proof of Theorem 2 is technical and not very intuitive; the new proof uses a direct combinatorial correspondence between these permutations and certain words determined by I. This simpler method applies universally, not just for "almost all" I. A promising next step would be to apply this to cycle types other than the full n-cycle, such as involutions (self-inverse permutations) or derangements (permutations with no fixed points). I had not made much progress with these before, but the correspondence with words may enable further insight.

Split graphs and combinatorial species. A *split graph* is a graph whose vertices can be partitioned into a clique (within which every pair is adjacent) and a stable set (within which no pair is adjacent). Figure 4 is an example of a split graph. These graphs have been of interest in graph theory for decades, and I noticed that certain problems in their enumeration are amenable to the methods of species theory, a powerful algebraic framework that treats combinatorial classes as algebraic objects to be operated on and combined in various ways. I have found several identities involving the species of split graphs, including:

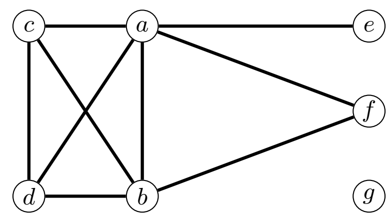

Figure 4: A split graph, with clique $\{a,b,c,d\}$ and stable set $\{e,f,g\}$.

▶▶▶ **Theorem 3.** (Troyka [10]) *If S is the species of split graphs,* BC *the species of bicolored graphs (i.e. bipartite graphs with a chosen 2-coloring), and L the species of linear orders, then* $\mathrm{BC} = L \cdot S$.

This theorem expresses an equality of species, which is a much stronger type of statement than the corresponding identity of generating functions $\mathrm{BC}(x) = \frac{1}{1-x}\,S(x)$; it signifies the existence of a natural equivalence between the two sides. I used Theorem 3 to prove a conjecture of Cheng, Collins, and Trenk [1] on the proportion of split graphs that are *balanced* (meaning the partition into a clique and a stable set is unique):

▶▶▶ **Theorem 4.** (Troyka [10]) *As $n \to \infty$, the proportion of split graphs on n vertices that are balanced goes to* 1.

That is, almost all split graphs are balanced.

I proved Theorem 3 using algebraic manipulation of the species involved; however, I would prefer to find a direct bijective proof. This problem would be ideal to work on with undergraduates.

References

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